

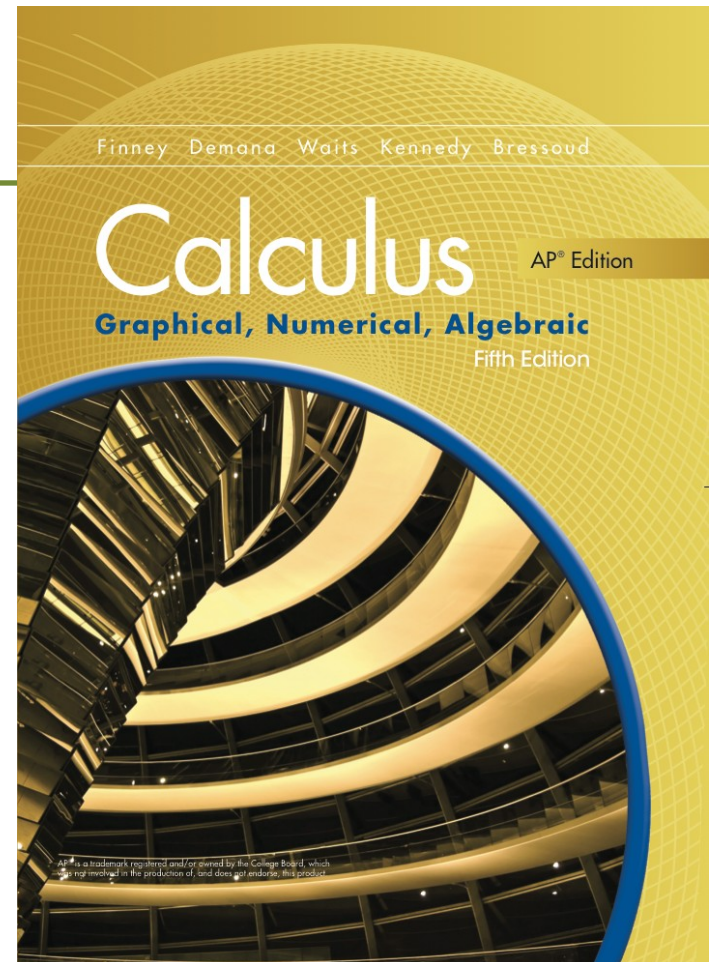
Chapter 7

Differential Equations and Mathematical Modeling



Section 7.5

Logistic Growth



Quick Review

1. Use polynomial division to write the rational function in the form $Q(x) + \frac{R(x)}{D(x)}$, where the degree of R is less than the degree of D .

a. $\frac{x^2 + x}{x - 1}$

b. $\frac{x^3 + 3}{x^2 - 1}$

Quick Review

2. Let $f(x) = \frac{30}{1 + 5e^{0.1x}}$.

- a. Find where $f(x)$ is continuous.
- b. Find $\lim_{x \rightarrow \infty} f(x)$.
- c. Find $\lim_{x \rightarrow -\infty} f(x)$.
- d. Find the y -intercept of the graph of f .
- e. Find all horizontal asymptotes of the graph of f .

Quick Review Solution

1. Use polynomial division to write the rational function in the form $Q(x) + \frac{R(x)}{D(x)}$, where the degree of R is less than the degree of D .

a. $\frac{x^2 + x}{x - 1}$ $x + 2 + \frac{2}{x - 1}$

b. $\frac{x^3 + 3}{x^2 - 1}$ $x + \frac{x + 3}{x^2 - 1}$

2. Let $f(x) = \frac{30}{1 + 5e^{0.1x}}$.

a. Find where $f(x)$ is continuous. $(-\infty, \infty)$

b. Find $\lim_{x \rightarrow \infty} f(x)$. 30

c. Find $\lim_{x \rightarrow -\infty} f(x)$. 0

d. Find the y -intercept of the graph of f . 5

e. Find all horizontal asymptotes of the graph of f . $y = 0$ and $y = 30$

What you'll learn about

- Logistic growth as a reasonable model for population growth
- The technique of partial fractions
- Solving a logistic differential equation by partial fractions
- Real-world applications of logistic growth
- The general logistic formula

... and why

Populations in the real world tend to grow logistically over extended periods of time.

Partial Fraction Decomposition with Distinct Linear Denominators

If $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with the degree of P less than the degree of Q , and if $Q(x)$ can be written as a product of distinct linear factors, then $f(x)$ can be written as a sum of rational functions with distinct linear denominators.

Example Finding a Partial Fraction Decomposition

Write the function $f(x) = \frac{6x^2 - 8x - 4}{(x^2 - 4)(x - 1)}$ as a sum of rational functions with linear denominators.

Since $f(x) = \frac{6x^2 - 8x - 4}{(x - 2)(x + 2)(x - 1)}$, we will find numbers A, B and C

so that $f(x) = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{C}{x - 1}$.

Note that $\frac{A}{x - 2} + \frac{B}{x + 2} + \frac{C}{x - 1} = \frac{A(x + 2)(x - 1) + B(x - 2)(x - 1) + C(x - 2)(x + 2)}{(x - 2)(x + 2)(x - 1)}$,

so it follows that $A(x + 2)(x - 1) + B(x - 2)(x - 1) + C(x - 2)(x + 2) = 6x^2 - 8x - 4$.

Setting $x = 2$: $A(4)(1) + B(0) + C(0) = 4$, so $A = 1$.

Setting $x = -2$: $A(0) + B(-4)(-3) + C(0) = 36$, so $B = 3$.

Setting $x = 1$: $A(0) + B(0) + C(-1)(3) = -6$, so $C = 2$.

Therefore $f(x) = \frac{6x^2 - 8x - 4}{(x - 2)(x + 2)(x - 1)} = \frac{1}{x - 2} + \frac{3}{x + 2} + \frac{2}{x - 1}$.

Example Antidifferentiating with Partial Fractions

Find $\int \frac{6x^2 - 8x - 4}{(x-2)(x+2)(x-1)} dx$

We know from the last example that

$$\begin{aligned}\int \frac{6x^2 - 8x - 4}{(x-2)(x+2)(x-1)} dx &= \int \left(\frac{1}{x-2} + \frac{3}{x+2} + \frac{2}{x-1} \right) dx \\ &= \ln|x-2| + 3\ln|x+2| + 2\ln|x-1| + C \\ &= \ln(|x-2||x+2|^3|x-1|^2) + C\end{aligned}$$

Logistic Differential Equation

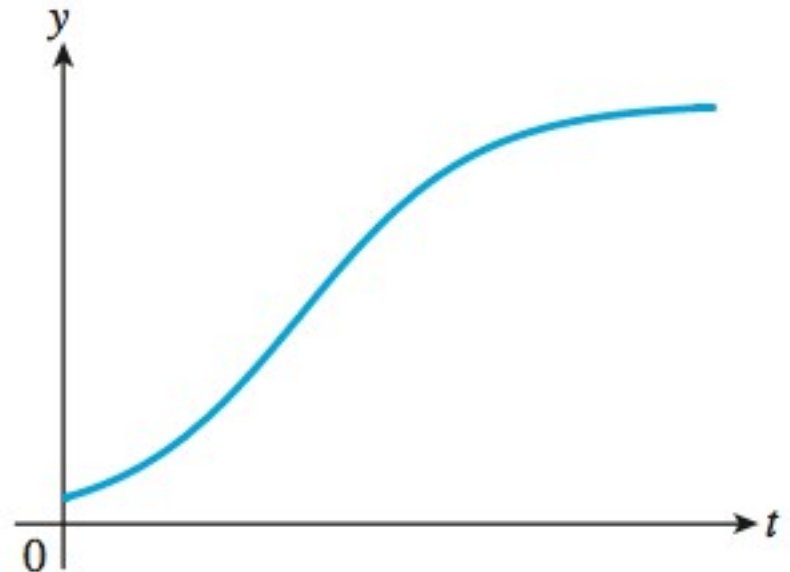
Exponential growth can be modeled by the differential equation

$$\frac{dP}{dt} = kP \text{ for some } k > 0.$$

If we want the growth rate to approach zero as P approaches a maximal carrying capacity M , we can introduce a limiting factor

of $M - P$:
$$\frac{dP}{dt} = kP(M - P).$$

This is the **logistic differential equation**.



Example Logistic Differential Equation

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation

$$\frac{dP}{dt} = 0.008P(100 - P), \text{ where } t \text{ is measured in years.}$$

- a. What is the carrying capacity for bears in this wildlife preserve?
 - b. What is the bear population when the population is growing the fastest?
 - c. What is the rate of change of the population when it is growing the fastest?
-
- a. The carrying capacity is 100 bears.
 - b. The bear population is growing the fastest when it is half the carrying capacity, 50 bears.
 - c. When $P = 50$, $\frac{dP}{dt} = 0.008(50)(100 - 50) = 20$ bears per year.

The General Logistic Formula

The solution of the general logistic differential equation

$$\frac{dP}{dt} = kP(M - P)$$

is

$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$

where A is a constant determined by an appropriate initial condition. The **carrying capacity** M and the **growth constant** k are positive constants.

Quick Quiz Sections 7.4 and 7.5

You may use a graphing calculator to solve the following problems.

1. The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time t . If there are 2 acres consumed when $t = 1$ and 3 acres consumed when $t = 5$, how many acres will be consumed when $t = 8$?

- (A) 3.750
- (B) 4.000
- (C) 4.066
- (D) 4.132
- (E) 4.600

Quick Quiz Sections 7.4 and 7.5

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Quick Quiz Sections 7.4 and 7.5

2. Let $F(x)$ be an antiderivative of $\cos(x^2)$. If $F(1) = 0$, then

$F(5) =$

(A) - 0.099

(B) - 0.153

(C) - 0.293

(D) - 0.992

(E) - 1.833

Quick Quiz Sections 7.4 and 7.5

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Quick Quiz Sections 7.4 and 7.5

3. $\int \frac{dx}{(x-1)(x+3)} =$

(A) $\frac{1}{4} \ln \left| \frac{x-1}{x+3} \right| + C$

(B) $\frac{1}{4} \ln \left| \frac{x+3}{x-1} \right| + C$

(C) $\frac{1}{2} \ln |(x-1)(x+3)| + C$

(D) $\frac{1}{2} \ln \left| \frac{2x+2}{(x-1)(x+3)} \right| + C$

(E) $\ln |(x-1)(x+3)| + C$

Quick Quiz Sections 7.4 and 7.5

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(E) $\ln |(x-1)(x+3)| + C$

Chapter Test

1. Evaluate $\int \frac{t dt}{t^2 + 5}$.
2. Evaluate $\int e^{3x} \sin x dx$
3. Evaluate $\int \frac{25}{x^2 - 25} dx$
4. Solve the initial value problem analytically: $\frac{dy}{dx} = y(1 - y)$, $y(0) = 0.1$.
5. Find an integral equation $y = \int_a^x f(t) dt$ such that $\frac{dy}{dx} = \sin^3 x$ and $y = 5$ when $x = 4$.

Chapter Test

6. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation $\frac{dL}{dx} = -kL$, where k is a constant.

As a diver you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below a tenth of the surface value. About how deep can you expect to work without artificial light?

7. Find the amount of time required for \$10,000 to double if the 6.3% annual interest is compounded **(a)** annually, **(b)** continuously.

Chapter Test Solutions

1. Evaluate $\int \frac{tdt}{t^2 + 5}$. $\frac{1}{2} \ln(t^2 + 5) + C$

2. Evaluate $\int e^{3x} \sin x dx$. $\left(\frac{3 \sin x}{10} - \frac{\cos x}{10} \right) e^{3x} + C$

3. Evaluate $\int \frac{25}{x^2 - 25} dx$. $\frac{5}{2} \ln \left| \frac{x-5}{x+5} \right| + C$

4. Solve the initial value problem analytically: $\frac{dy}{dx} = y(1 - y)$, $y(0) = 0.1$.

$$y = \frac{1}{1 + 9e^x}$$

5. Find an integral equation $y = \int_a^x f(t) dt$ such that $\frac{dy}{dx} = \sin^3 x$ and $y = 5$

when $x = 4$. $y = \int_4^x \sin^3 t dt + 5$

Chapter Test Solutions

6. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation $\frac{dL}{dx} = -kL$, where k is a constant.

As a diver you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below a tenth of the surface value. About how deep can you expect to work without artificial light?

about 59.8 feet

7. Find the amount of time required for \$10,000 to double if the 6.3% annual interest is compounded **(a)** annually, **(b)** continuously.

a. about 11.3 years

b. about 11 years